VALUES FOR TEAMS GAMES

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1. Introduction

In certain applications it is interesting to consider cooperative games that are not necessarily defined for every coalition $S \subset N$ (or equivalently, whose values on certain coalitions are not relevant for the corresponding problem). For example, if we think of N as the set of all players in the NBA and $v(S)$ is the worth of a basketball team "on the floor", then it only makes sense to consider $v(S)$ for coalitions of cardinality 5. With this and similar examples in mind, we define a team game as a game that vanishes on all coalitions but those of a certain fixed cardinality s.

We assume that there is a given, exogenous amount c to be distributed among the players. Continuing with the NBA example, we could think that c is a certain amount coming, say, from television rights for game transmission which has been agreed to distribute among the players. We propose several solutions satisfying certain desired properties. Moreover, these solutions could also be seen as an alternative proposal to establish players salaries and form teams using as criterium that of having a more competitive league.

More explicitly, the results obtained in this article are: First, a concrete expression for every linear symmetric solution (in fact, the dimension of all such solutions is found to be three); as a corollary we get also formulas for solutions that further satisfy either the efficiency or the naturalness axioms. Finally, it is shown that there exists a unique linear solution which is symmetric, natural and efficient.

The article ends by studying bankruptcy games as a particular case of these type of games.

2. General results

In this section we take up the problem of dividing a sum c in a "fair" way, among the players in a team game. One wants that each player's share depends on his performance, although we take into account only the teams' performance.

As an example, we could think that a certain company is organized into work teams, and that every project that arrives is assigned to one and only one of these teams. Individual members of the company can participate in several different teams, which we assume always have the same number of participants.

The problem then is how to divide, among the workers, a certain profit obtained from these projects considering only the performance of the teams. Let G be the set of TU cooperative games with space of players $N = \{1, \ldots, n\}$.

Definition 1. *For every $s : 1, \ldots, n$, let $G_s = \{v \in G \mid \ v(S) = 0 \text{ if } |S| \neq s\}$; we think of the elements in G_s as "team games" with s players.*

We look at pairs $(v, c) \in G_s \oplus \mathbb{R}$ and solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ that are linear and symmetric. Here $1 \leq s < n$.

Every $v \in G_s$ can be (uniquely) written as $v = x^s + w$, where $x \in \mathbb{R}^n$ and $w \in W_s$.

Lemma 1. *With the previous notation, the vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ may be computed from v using the following formula*

$$x_i = \frac{\sum_S v(S)}{s \begin{pmatrix} n \\ s \end{pmatrix}} + \frac{s(n-s)}{n \begin{pmatrix} n-2 \\ s-1 \end{pmatrix}} \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s} \right].$$

Remark 1. *Using Shapley's value one can give an alternate (equivalent) description of the x^s component of an arbitrary team game $v \in G_s$. Namely, if $v = k{1_n}^s + z^s + w$ is the decomposition of v with*

$$k = \frac{\langle v, (1_n)^s \rangle}{\langle (1_n)^s, (1_n)^s \rangle} = \frac{\sum_{S \subset N} v(S)}{s \begin{pmatrix} n \\ s \end{pmatrix}}$$

and $z \in \Delta_n^{\perp}$, then

$$Sh(v) = Sh(z^s) = \frac{1}{n-1} z,$$

since $Sh(w) = 0$ by Schur and $Sh({1_n}^s) = 0$, $s < n$, because of efficiency.

Therefore,

$$v = [k(1_n) + (n-1)Sh(v)]^s + w \in (\mathbb{R}^n)^s \oplus W_s$$

is the decomposition of any team game $v \in G_s$.

Definition 2. *For $x \in \mathbb{R}^n$ define*

$$x(N) := \sum_{i=1}^{n} x_i.$$

Proposition 1. *The space of linear, symmetric solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is 3-dimensional. Moreover, their general expression is given by*

$$\phi(x^s + w, c) = \left[\alpha \frac{x(N)}{n} + \gamma c \right] 1_n + \beta (x - \frac{x(N)}{n} 1_n) \tag{1}$$

for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$. Equivalently,

$$\phi_i(v, c) = Ac + B \sum_{S \ni i} v(S) - C \sum_{S \not\ni i} v(S)$$

for every $(v, c) \in G_s \oplus \mathbb{R}$, $i : 1, \ldots, n$, for arbitrary $A, B, C \in \mathbb{R}$.

Definition 3. *(Efficiency Axiom) The solution $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is said to be efficient if*

$$\phi(v, c) \cdot 1_n = c.$$

Theorem 1. *The space of linear, symmetric solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ that are also efficient is 1-dimensional. Their general expression is given by*

$$\phi(x^s + w, c) = \frac{c}{n} 1_n + \beta(x - \frac{x(N)}{n} 1_n)$$

for arbitrary $\beta \in \mathbb{R}$.

An equivalent formulation to the above is that every linear symmetric efficient solution is of the form

$$\phi_i(v, c) = \frac{c}{n} + \lambda \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n - s} \right]$$

for every $(v, c) \in G_s \oplus \mathbb{R}$, $i : 1, \ldots, n$, for arbitrary $\lambda \in \mathbb{R}$.

Definition 4. *(Naturalness Axiom) The solution $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ is said to be natural if*

$$\phi(x^s, x \cdot 1_n) = x.$$

for every x.

Proposition 2. *The space of linear, symmetric solutions $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ that are also natural is 1-dimensional. Their general expression is given by*

$$\phi(x^s + w, c) = \left[\alpha \frac{x(N)}{n} + \frac{1 - \alpha}{n} c \right] 1_n + (x - \frac{x(N)}{n} 1_n) = x + \frac{(1 - \alpha)}{n} (c - x \cdot 1_n) 1_n$$

for arbitrary $\alpha \in \mathbb{R}$.

Corollary 1. *There exists a unique solution $\psi : G_s \oplus \mathbb{R} \to \mathbb{R}^n$ which is linear, symmetric, efficient and natural. It is given by*

$$\psi(x^s + w, c) = x + \frac{1}{n} (c - x \cdot 1_n) 1_n.$$

Theorem 2. *The formula for the unique linear, symmetric, efficient and natural solution, ψ, on $G_s \oplus \mathbb{R}$ takes the form:*

$$\psi(v, c) = \frac{c}{n} 1_n + (n - 1) Sh(v).$$

Remark 2. *This unique solution can also be written as*

$$\psi_i(v, c) = \frac{c}{n} + \frac{n - 1}{\binom{n}{s}} \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n - s} \right]$$

for every $(v, c) \in G_s \oplus \mathbb{R}$, $i : 1, \ldots, n$.

Example. When a traveler purchases a Europass he can choose to visit a subset of s countries (where $s = 3, 4$ or 5) from a set of N European countries, to travel for a certain fix amount of days d. We could think of a game (v, c) where $v \in G_s$ is such that $v(S) =$ the number of travelers (in a particular summer, say) that chose the set S of countries for his pass, and where c is the total amount collected, at the end of the period, from Europasses purchased.

Then a solution $\phi : G_s \oplus \mathbb{R} \to \mathbb{R}^N$ will assign to (v, c) the vector $\phi(v, c)$ where $\phi_j(v, c)$ is the amount that corresponds to country j.

3. Bankruptcy problem

Recall that G_1 consists of games that can only take non zero values on cardinality one subsets of N. Thus, it is naturally identified with $\mathbb{R}^n$: $x(\{i\}) = x_i$.

Definition 5. *A bankruptcy game is an element* $(x, c) \in G_1 \oplus \mathbb{R}$ *such that*

$$x(N) = \sum_{i=1}^{n} x_i \geq c.$$

We interpret x_i as the amount that the ith creditor demands, whereas c is the total amount that may be repaid.

All the results of the previous section apply to bankruptcy games. Let us summarize them together with an interpretation of their meaning.

- There is a 3-dimensional space of solutions $\phi : G_1 \oplus \mathbb{R} \to \mathbb{R}^n$ for the bankruptcy problem.
- Efficiency (i.e., $\phi(x, c) \cdot 1_n = c$) means that the value of all the remaining goods, c, are divided among all creditors. There is a 1-dimensional space of efficient solutions given by
$$\phi(x, c) = \frac{c}{n} 1_n + \lambda(x - \frac{x(N)}{n} 1_n)$$
for arbitrary $\lambda \in \mathbb{R}$.
- Naturalness (i.e., $\phi(x, x \cdot 1_n) = x$) is the axiom that says that if the total amount claimed by the creditors equals the value of the remaining goods, then each creditor receives what he claims. There is a 1-dimensional subspace of natural solutions to the bankruptcy problem. They are given by
$$\phi(x, c) = x + \frac{(1 - \alpha)}{n} (c - x \cdot 1_n) 1_n$$
for arbitrary $\alpha \in \mathbb{R}$.

Finally,

Corollary 2. *There exists a unique bankruptcy solution which is linear, symmetric, efficient and natural. It is given by*

$$\phi(x, c) = x + \frac{1}{n}(c - x \cdot 1_n) 1_n = \frac{c}{n} 1_n + (n - 1) Sh(v).$$

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